

INVARIANT POINTS OF A SURFACE TRANSFORMATION OF GIVEN CLASS*

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1. **Introduction.** The theory of the invariant points of a surface transformation is not only of interest in itself but of some importance in its applications to dynamical and other problems. As such, it has already received a certain amount of attention. Thus, for example, Brouwer has studied the transformations of a sphere and of a projective plane, Kerékjártó the transformations of a sphere and of a plane ring, Nielsen and Brouwer the transformations of the anchor ring and of the non-orientable surface of the same connectivity.† In looking for periodic orbits of dynamical systems, Birkhoff has shown that if a closed orientable surface of genus p be carried into itself by a one-to-one analytic transformation, the difference between the number of directly unstable and other invariant points is a function of the class of the transformation alone and has the value $2p - 2$ when the transformation is of the class of the identity, that is to say, when the transformation can be generated by a continuous deformation of the surface.

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† L. E. J. Brouwer, *Über eindeutige stetige Transformationen von Flächen in sich*, *Mathematische Annalen*, vol. 69 (1910), pp. 176-180.

B. von Kerékjártó, *Über die Brouwerschen Fixpunktsätze*, *Mathematische Annalen*, vol. 80 (1919), pp. 29-32; *Über Transformationen des ebenen Kreisringes*, *Mathematische Annalen*, vol. 80 (1919), pp. 33-35.

J. Nielsen, *Über die Minimalzahl der Fixpunkte bei den Abbildungstypen der Ringflächen*, *Mathematische Annalen*, vol. 82 (1921), pp. 83-93.

L. E. J. Brouwer, *Über die Minimalzahl der Fixpunkte bei den Klassen von eindeutigen stetigen Transformationen der Ringflächen*, *Mathematische Annalen*, vol. 82 (1921), pp. 94-96.

See also the note *Sur quelques applications de l'indice de Kronecker*, by Hadamard appended to Tannery's *Introduction à la Théorie des Fonctions d'une Variable*, and the following:

J. W. Alexander, *On transformations with invariant points*, these *Transactions*, vol. 23 (1922), pp. 89-95.

G. D. Birkhoff and O. D. Kellogg, *Invariant points in function space*, these *Transactions*, vol. 23 (1922), pp. 96-115.

G. D. Birkhoff, *Dynamical systems with two degrees of freedom*, these *Transactions*, vol. 18 (1917), pp. 199-300. In particular, pp. 286 et seq.

I shall consider in this paper a general s -to-1 continuous transformation of a surface into itself and obtain, (§ 6), the explicit relation between the sum of the indices of the invariant points, (§ 2), and the class of the transformation, (§ 3). The generalization from a 1-to-1 to an s -to-1 transformation is only significant when the connectivity of the surface does not exceed 3, (§ 3).

So as not to lengthen the paper unduly, I have assumed with only a few words of explanation some of the simpler topological properties of surfaces. The elements of two-dimensional analysis situs are sufficiently well established by now to permit of their free use without too great insistence on details.

2. Index of a curve and of an invariant point. Let Δ be a continuous transformation of the plane or of a portion of the plane such that it carries a finite region R into a finite region \bar{R} and a variable point P of R into a variable point \bar{P} of \bar{R} . Then, if P is not an invariant point of the transformation, the angle θ made by the vector $P\bar{P}$ with the positive X -axis is determined in value except for an additive constant of the form $2\tau\pi$ and varies continuously as the point P describes a path in R , provided, of course, that the path avoids the invariant points of Δ , at which θ ceases to have any meaning. Now, if the point P describes a closed path of the above sort in a definite sense, the final value of θ as P returns to its original position will differ from the initial value of θ by an integral multiple τ of 2π , at most. We shall call this number τ the *index* of the closed sensed path with respect to the transformation Δ . This index obviously remains unaltered under any continuous deformation of the path which avoids the invariant points of Δ ; consequently, if such a deformation reduces the path to a (non-invariant) point, the index of the path must be zero. We may mention, in passing, that if A be a closed sensed path and \bar{A} its image under a transformation Δ , the index of A with respect to Δ is the same, — not the negative of, — the index of \bar{A} with respect to the inverse transformation Δ^{-1} . Under a change of coördinates in the plane, the index of a sensed curve either remains invariant or changes in sign according as the orientation of the new system is the same as or the reverse of that of the old.

The *index* of an isolated invariant point $F = \bar{F}$ of Δ is defined as the index of the boundary B , positively described, of any simply connected region containing F but no other invariant point of Δ . The index of an invariant point is completely independent of the coördinate system. To be sure, the index of the sensed curve B changes sign when we pass from a right-handed to a left-handed system of coördinates, but the positive direction along B is likewise reversed, so that the two changes exactly compensate one another. From this, it follows that a meaning may be given to the index of an isolated invariant point F of any continuous surface transformation, not necessarily planar, provided the surface undergoing the transformation is free from singularities

in the neighborhood of the fixed point F . For there is no difference, topologically, between the neighborhood of a simple point of a surface and the neighborhood of a point in a plane. It is easy to verify that the index of an invariant point F is the same both with respect to the direct transformation and its inverse Δ^{-1} .

Returning to the plane, let us consider any closed region R' in R bounded by a curve B' and such that there are at most a finite number of invariant points of Δ in R' , none of which are on the curve B' . Then the index of the curve B' described in the positive sense evidently measures the sum of the indices of the invariant points within the region R' , as may be seen at once by subdividing the region R' into simply connected pieces such that no piece contains more than one invariant point and that no invariant point is on the common boundary of two or more pieces. For, from its definition, the index of B' is equal to the sum of the indices of the boundaries of all the component parts of R' determined by the subdivision.

3. Surface transformations; class and index of a transformation.

Let S be a closed surface of connectivity κ , and let Δ be a continuous transformation which carries a variable point P of S into a variable point \bar{P} of the same surface.* The inverse transformation Δ^{-1} need not be single valued, but we shall assume that, with a finite number of exceptions, it carries a point \bar{P} back into s distinct points of the surface S , which simply means that the transformation Δ is of the so-called s -to-1 type. On the manifold $\bar{S} (\equiv S)$ of the transformed points \bar{P} , let us form a covering surface S^* of s layers such that the correspondence Δ^* from S to S^* associated with the correspondence Δ from S to \bar{S} is everywhere 1-to-1. Then, by an immediate application of Euler's formula

$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - \kappa,$$

we may compare the connectivity κ^* of the covering surface S^* with the connectivity $\bar{\kappa}$ of the surface \bar{S} , and so obtain the relation

$$(1) \quad 3 - \kappa^* = s(3 - \bar{\kappa}) - b,$$

where b is the number of equivalent simple branch points of the covering surface. But the surfaces S^* and \bar{S} are each images of the surface S ; therefore, $\kappa^* = \bar{\kappa} = \kappa$, and (1) reduces to

* When the surface is orientable of genus p , we have $p = \frac{1}{2}(\kappa - 1)$.

$$(2) \quad (s-1)(3-\kappa)-b=0.$$

From (2), we conclude that if κ exceeds 3, the transformation must be 1-to-1, and that if κ is equal to 3 the covering surface S^* can have no branch points.

If Δ be an s -to-1 transformation of an oriented surface S into itself, the number $\tau = \pm s$ will be defined as the *index* of the transformation, where the positive or the negative sign is to be chosen before s according as the transformation preserves or reverses orientation. Two transformations of a surface S into itself will be said to belong to the same *class* if and only if they differ from one another by a continuous deformation of the surface into itself. The latter definition applies whether the surface S be orientable or not.

4. Surfaces of genus 0. By way of introduction to the general problem, let us first examine an s -to-1 transformation of a closed surface S of genus 0 into itself, although we shall obtain nothing new in this case. We shall assume that the transformation Δ possesses at most a finite number of invariant points. So far as generality is concerned, it is worth bearing in mind that an unrestricted s -to-1 transformation may be regarded as a limiting case of a transformation of type Δ involving a parameter λ .

We represent the surface S by the complex plane and so select the point at infinity that it is neither an invariant point of the transformation Δ nor a point at which the covering surface S^* of § 3 has a branch point. There will then be s distinct finite points P_1, P_2, \dots, P_s of the plane which are carried to infinity by the transformation Δ . Moreover, if indices be assigned to these points, just as if they were invariant points, it will be found that the index of each is -1 or $+1$, according as the transformation preserves or reverses orientation, because a small positively sensed circle about a point P_i will be carried into a distant curve about infinity encircling the point P_i once in the negative or positive sense according as the transformation preserves orientation or not. In general, there will also be certain fixed points F_j with indices τ_j respectively.

Now, let C be a positively sensed circle with center at the origin and large enough to enclose all the points P_i and F_j . Then, by § 2, the index of C will measure the sum of the indices of all the points P_i and F_j together. Moreover, if the circle C be sufficiently large, its image will be a small curve entirely within C and in the neighborhood of the image of the point at infinity under the transformation Δ . The index of the curve C will, therefore, be unity, and we shall have

$$1 = \sum_j \tau_j \mp s$$

where the sign before s depends upon whether the transformation preserves orientation or not. Thus, if $\tau = \pm s$ be the index of the transformation, (§ 3), we shall have the relation

$$(3) \quad \sum_j \tau_j = 1 + \tau.$$

This is Brouwer's theorem in a generalized form. In the paper referred to in the footnote to § 1, I have given another proof of this theorem, so framed as to generalize to n dimensions and to require no restrictions whatever on the character of the inverse transformation Δ^{-1} .*

The above argument is very closely related to one of the well known proofs of the fundamental theorem of algebra.† In fact, the roots of an equation $f(z) = 0$ are the fixed points of the transformation

$$\bar{z} = z + f(z),$$

and conversely.

5. Intersection numbers. To *orient* an orientable surface S , we choose two sensed arcs X and Y crossing one another at a point P of S and arbitrarily agree to say that X makes a positive crossing with Y at P and Y a negative crossing with X . Positive and negative crossings are then determined at all points of S by continuity considerations. Moreover, with any two closed sensed curves A and B of the oriented surface S , there is associated a so-called *intersection number*

$$N(AB) = -N(BA)$$

which can be interpreted to measure the difference between the number of times the curve A makes a positive crossing with the curve B and the number of times it makes a negative crossing with B . This method of interpretation implies, of course, that the curve A crosses the curve B a finite number of times only. However, the number $N(AB)$ may be defined by continuity considerations whether or not this condition is actually realized. The number $N(AB)$ is invariant under continuous deformations of A and B on the surface S .

6. The fundamental relation. Let us now consider the case of a 1-to-1 transformation Δ of a closed oriented surface S into itself, where the genus p of S is greater than zero. Through a point C of the surface S , we choose a canonical system of A and B curves

$$(K) \quad A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_p,$$

* See also the very important generalizations to function space by Birkhoff and Kellogg, loc. cit.

† Cf. Osgood, *Lehrbuch der Funktionentheorie*, pp. 219 et seq.

sensed in such a manner that $N(A_i B_i) = -N(B_i A_i) = 1$ ($i = 1, 2, \dots, p$). Then, under the transformation Δ , the point C is carried into a point \bar{C} , and the system (K) into a system

$$(\bar{K}) \quad \bar{A}_1, \bar{A}_2, \dots, \bar{A}_p; \bar{B}_1, \bar{B}_2, \dots, \bar{B}_p.$$

Moreover, $N(\bar{A}_i \bar{B}_i) = -N(\bar{B}_i \bar{A}_i) = 1$, since the transformation preserves orientation. We shall assume that the transformation Δ possesses at most a finite number of invariant points and that the system (K) may be so chosen (i) as to pass through no invariant point and (ii) as to meet the system (\bar{K}) in at most a finite number of points at each of which a curve of (K) definitely crosses a curve of (\bar{K}) . Actually, these conditions are always realized either for Δ itself or for some arbitrarily neighboring transformation Δ' of the same class as Δ . The formula that we are about to prove is the following generalization of (3):

$$(4) \quad \sum_j \tau_j = 1 + \tau + \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}$$

where the transformation index τ is unity in this particular case, since we are assuming that Δ is 1-to-1 and preserves orientation. If the class of Δ contains the identity,

$$N(B_i \bar{A}_i) = N(\bar{B}_i A_i) = N(B_i A_i) = -1,$$

and (4) reduces to

$$\sum_j \tau_j = 2 - 2p,$$

from which Professor Birkhoff's theorem, mentioned in the introduction, may at once be deduced.

To prove relation (4), we first cut the surface S along the transformed canonical curves (\bar{K}) and thereby resolve it into a simple polygonal region of $4p$ sides which we represent by the interior of a circle \bar{K} in the plane. The circumference of the circle \bar{K} now represents the system (\bar{K}) itself. It consists of $4p$ arcs following one another in the canonical order

$$\bar{A}_1 \bar{B}_1 \bar{A}_1^{-1} \bar{B}_1^{-1} \dots \bar{A}_p \bar{B}_p \bar{A}_p^{-1} \bar{B}_p^{-1}$$

as the circle is described in the positive sense. The original canonical system (K) is represented within the circle \bar{K} by a system of arcs terminating either within \bar{K} and at the point C or on the circumference \bar{K} itself. Two or more

of these arcs will always be needed to represent an entire curve of the system (K) . The situation is as illustrated by Figure 1 which represents the case of a relatively simple transformation of a surface of genus 2.

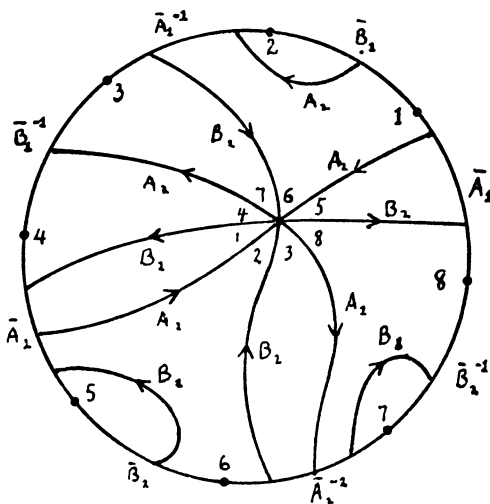


Fig. 1.

Now, it will be observed that the system of arcs (K) subdivides the interior of the circle \bar{K} into a finite number of simply connected regions R_i , and that each of the regions R_i is carried by the transformation Δ into a region \bar{R}_i such that \bar{R}_i is not cut by any of the curves (\bar{K}) on S and is therefore represented in the plane by a single connected piece. Hence, in the plane, we may speak about the index of the boundary of each region R_i . Moreover, by § 2, the sum σ_0 of the indices of the boundaries of all the regions R_i , where each boundary is described in the positive sense, is equal to the sum of the indices of the invariant points of Δ :

$$(5) \quad \sigma_0 = \sum_j \tau_j.$$

It is perhaps superfluous to observe that the transformations of the individual regions R_i of the plane do not combine to form one single continuous transformation of the interior of the circle \bar{K} into itself, so that it would be quite meaningless to say that the sum σ_0 was equal to the index of the circle \bar{K} . The index of \bar{K} is not defined at all.

We must next identify the sum σ_0 with the right-hand member of (4) in order to establish the validity of that relation. Now, the number $2\pi\sigma_0$ measures

the total variation in the direction θ of a vector $P\bar{P}$ joining a point P within or on the circle \bar{K} to its image \bar{P} , as the point P describes the boundaries of all the regions R_i in the positive sense. Moreover, these boundaries consist, in part, of arcs of the system (K) and, in part, of arcs of the circle \bar{K} , so that we may write

$$2\pi\sigma_0 = 2\pi\sigma + 2\pi\bar{\sigma},$$

where the two expressions on the right measure the contribution to the total variation of θ coming from the arcs of (K) and of \bar{K} respectively.

To make matters definite, let us first evaluate the contribution to the variation $2\pi\sigma$ from a single curve of (K) , such as A_1 . In spite of the fact that every arc of A_1 is traversed exactly twice in opposite directions as the boundaries of the regions R_i are described, the total variation of θ over A_1 is not zero, for the image of a point P of A_1 is to be taken on the arc \bar{A}_1 of \bar{K} when A_1 is being described in the direct sense, but on the arc \bar{A}_1^{-1} of \bar{K} when A_1 is being described in the reverse sense, corresponding to the fact that the regions on the two sides of an arc of A_1 are mapped upon regions contiguous to \bar{A}_1 and \bar{A}_1^{-1} respectively. Let us denote the images of the point P of A_1 on \bar{A}_1 and \bar{A}_1^{-1} by \bar{P} and \bar{P}^{-1} respectively. Then, the sum of the variations in direction of $P\bar{P}$ and $P\bar{P}^{-1}$ as P describes A_1 in the direct and reverse senses respectively is equal to the difference of these variations as P is described in the direct sense alone, which difference is merely the variation of the angle $\bar{P}^{-1}P\bar{P}$ denoted by ψ in Figure 2.

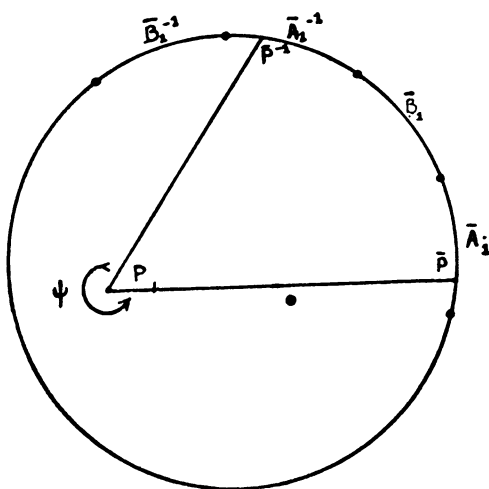


Fig. 2.

Now, as the point P describes the planar representation of the curve A_1 , it makes a sudden leap from one point of the circle \bar{K} to another whenever it passes from one arc of A_1 to the next. Moreover, when such a leap occurs, we must be prepared for an abrupt variation in the angle ψ , which variation is not to be measured in computing the contribution from A_1 to the variation $2\pi\sigma$. Actually, however, by the elementary properties of angles inscribed in a circle, the angle ψ undergoes no variation at a leap of the point P unless the leap is from one to the other of the two arcs of \bar{K} determined by the points \bar{P} and \bar{P}^{-1} , when the angle ψ changes by $\pm\pi$, depending on the direction of the leap. We conclude immediately that ψ decreases by π whenever the point P jumps from the arc \bar{B}_1 to the arc \bar{B}_1^{-1} of \bar{K} , increases by π whenever P jumps from \bar{B}_1^{-1} to \bar{B}_1 , but suffers no discontinuity at any other jump.

Let us now denote by α_1 and α'_1 the positive angles subtended at the point C by the arcs \bar{A}_1 and \bar{A}_1^{-1} of \bar{K} respectively. The total variation of ψ over A_1 , including all the jumps, is then $\alpha_1 + \alpha'_1$, since the point P starts from and returns to C , whereas the points \bar{P} and \bar{P}^{-1} describe \bar{A}_1 and \bar{A}_1^{-1} respectively in such a way as to augment ψ by α_1 and α'_1 respectively. The effective variation of ψ , not counting the jumps, is, therefore,

$$\alpha_1 + \alpha'_1 - \pi N(A_1 \bar{B}_1),$$

as may be seen at once by recalling the significance of the intersection number $N(A_1 B_1)$. In similar fashion, the contribution to the variation $2\pi\sigma$ from the curve A_i is

$$(6) \quad \alpha_i + \alpha'_i - \pi N(A_i \bar{B}_i)$$

while that from the curve B_i is

$$(7) \quad \beta_i + \beta'_i + \pi N(B_i \bar{A}_i),$$

where the meaning of β_i and β'_i is similar to that of α_i and α'_i . Thus adding (6) and (7) and summing with respect to i , we finally obtain

$$(8) \quad 2\pi\sigma = 2\pi + \pi \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

In deriving (8), we have, of course, made use of the relations

$$\sum (\alpha_i + \alpha'_i + \beta_i + \beta'_i) = 2\pi$$

and

$$-N(A_i \bar{B}_i) = N(\bar{B}_i A_i).$$

The variation $2\pi\bar{\sigma}$ is computed in similar fashion. Let the system

$$(k) \quad a_1, a_2, \dots, a_p; \quad b_1, b_2, \dots, b_p$$

be the image of the system (\bar{K}) under the transformation Δ . We then obtain without difficulty the relation

$$(9) \quad 2\pi\bar{\sigma} = 2\pi + \pi \sum_i \{N(\bar{B}_i a_i) + N(b_i \bar{A}_i)\}.$$

Now,

$$N(\bar{B}_i a_i) = N(B_i \bar{A}_i) \text{ and } N(b_i \bar{A}_i) = N(\bar{B}_i A_i).$$

Therefore, from (8) and (9), we obtain

$$(10) \quad \sigma_0 = \sigma + \bar{\sigma} = 2 + \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

Relation (4) now follows at once from (5) and (10), since $\tau = 1$.

7. Extension to a general transformation. Without going into details, let us merely indicate where the analysis of § 6 varies if it is assumed that the transformation Δ reverses orientation instead of preserving it.

(i) When cuts are made along the system (\bar{K}) and the resulting polygon is mapped in the plane in such a way that a positive crossing on the surface S is represented by a positive crossing in the plane, we find that, in this case, the arcs of the circle \bar{K} follow one another in the cyclic order

$$\bar{B}_p \bar{A}_p \bar{B}_p^{-1} \bar{A}_p^{-1} \dots \bar{B}_1 \bar{A}_1 \bar{B}_1^{-1} \bar{A}_1^{-1}$$

as \bar{K} is described in the positive sense.

(ii) As a result of the above change, the contributions to $2\pi\sigma$ from the curves A_i and B_i become

$$-\alpha_i - \alpha'_i - \pi N(A_i \bar{B}_i)$$

and

$$-\beta_i - \beta'_i + \pi N(B_i \bar{A}_i)$$

respectively, giving

$$2\pi\sigma = -2\pi + \pi \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

(iii) The expression for $2\pi\bar{\sigma}$ is now

$$(9') \quad 2\pi\bar{\sigma} = 2\pi - \pi \sum_i \{N(\bar{B}_i a_i) + N(b_i \bar{A}_i)\}.$$

However, since the transformation reverses orientation,

$$N(\bar{B}_i a_i) = -N(B_i \bar{A}_i), \quad N(b_i \bar{A}_i) = -N(\bar{B}_i A_i),$$

so that

$$2\pi\bar{\sigma} = 2\pi + \pi \sum \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

Thus, in place of (10), we have

$$(10') \quad \sigma_0 = \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

Thus, by (5) and (10'), relation (4) is again verified, since τ is now equal to -1 .

To complete the discussion for an orientable surface S , there remains to be considered the case of an s -to-1 transformation, where s is greater than unity. However, by § 3, a transformation of this sort can only exist if the genus p of the surface is unity,

$$p = \frac{1}{2}(\kappa - 1) = 1,$$

so that we are back to the case already discussed by Nielsen and Brouwer.* It will, therefore, be sufficient to verify relation (4) in terms of the results obtained by these mathematicians. We merely remark, in passing, that the method of § 6 may readily be extended to cover this case also. We take as the system (\bar{K}) , along which the cuts are made, a pair of canonical curves \bar{A} and \bar{B} , each of which is the image of s distinct curves $A^{(1)}, \dots, A^{(s)}$ and $B^{(1)}, \dots, B^{(s)}$. These $2s$ curves form the system (K) . The system (k) , however, consists of two curves a and b only. Proceeding as before, we find

$$2\pi\sigma = 2\pi\tau + \pi\{N(B\bar{A}) + N(\bar{B}A)\},$$

$$2\pi\bar{\sigma} = 2\pi + \pi\{N(B\bar{A}) + N(\bar{B}A)\},$$

therefore,

$$(4') \quad \sum_j \tau_j = 1 + \tau + N(B\bar{A}) + N(\bar{B}A).$$

This is nothing but relation (4) for the special case under consideration.

The results of Nielsen and Brouwer may be put in the following form. If the curves A and B transform into

$$\bar{A} = A^m B^n \text{ and } \bar{B} = A^p B^q$$

respectively, then

$$(4'') \quad \sum \tau_q = mq - np - m - q + 1.$$

Relations (4') and (4'') are equivalent, however, for

* Loc. cit., *Mathematische Annalen*, vol. 82.

$$N(B\bar{A}) = -m, \quad N(\bar{B}A) = -q,$$

$$\tau = mq - np.$$

8. Non-orientable surfaces. The case of a non-orientable or so-called "one-sided" surface is much less important in the applications and may be dealt with in summary fashion as follows.

With every non-orientable surface T , there is associated a unique orientable 2-sheeted covering surface S such that each point P_s of S corresponds to the point P_t of T which it covers associated with one of the two possible choices of an indicatrix at P_t . Moreover, every transformation Δ of the surface T into itself determines two transformations $\Delta^{(1)}$ and $\Delta^{(2)}$ of the covering surface S such that the two points $P_s^{(1)}$ and $P_s^{(2)}$ of S covering a point P_t of T are transformed respectively into the two points covering the image of P_t under Δ . The two transformations $\Delta^{(1)}$ and $\Delta^{(2)}$ differ merely by a permutation of the sheets of S ; one of them preserves, the other reverses orientation. Finally, every invariant point of Δ corresponds to an invariant point of one but not both of the transformations $\Delta^{(1)}$ and $\Delta^{(2)}$. Thus, the sum of the indices of the fixed points of Δ is expressed as the sum of a similar pair of sums for $\Delta^{(1)}$ and $\Delta^{(2)}$, and we are back to the orientable case already considered.

9. Boundaries. In the applications to dynamics,* it is useful to consider the transformations Δ of an orientable surface S of given genus but bounded by d simple closed curves, each of which is transformed into itself in such a way as to leave no point invariant. Obviously, each of these curves plays the rôle of an invariant point of index unity. For if the surface S be closed by the addition of d simply connected pieces bounded by the d curves respectively, it will be possible to extend the definition of the transformation Δ to the added pieces in such a way that within each piece there will be a single invariant point of index unity. It would also be possible to consider the case where two or more boundaries were interchanged. The presence of such boundaries would not affect the fundamental formula at all. To sum up, if the surface S contains boundaries of which d are transformed into themselves without invariant point,

$$\sum_j \tau_j + d = 1 + \tau + \sum_{i=1}^r \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

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* Cf. Birkhoff, loc. cit., pp. 290, et seq.